Exercise Set #10

"Discrete Mathematics" (2025)

Exercise 7 is to be submitted on Moodle before 23:59 on May 5th, 2025

Note: Spanning trees exist only for connected graphs. All graphs in this exercise set are to be assumed as connected.

E1. Let G be a graph and $L(G)$ be the Laplace matrix. If $L_0(G)$ is the matrix obtained by removing the last row and column of $L(G)$, show that

$$\det L_0(G) = \frac{1}{n}\lambda_1\lambda_2 \dots \lambda_{n-1}$$

where $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ are the eigenvalues of $L(G)$ with multiplicities (could be possibly zero). **Hint:** $\det(L(G) - xI_n) = -x(\lambda_1 - x)(\lambda_2 - x)\dots(\lambda_{n-1} - x)$.

E2. Find the number of spanning trees of the following graphs.

(1) The complete graph $K_n = \left([n], \binom{[n]}{2}\right)$. That is to say that K_n is a graph with n vertices such that there is an edge between any two vertices.

(2) The complete bipartite graph $K_{n,m} = ([n] \sqcup [m], [n] \times [m])$. That is, the vertices are into two groups of size n and m and there is an edge between each vertex of one group to another.

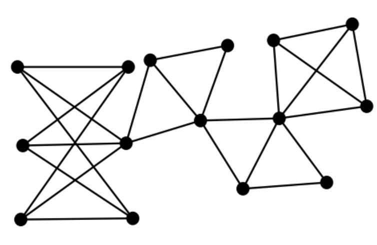
(3) A path graph $P_n = ([n], E)$ where $E = \{\{i, i+1\}\}_{i=1}^{n-1}$.

(4) A circular graph $C_n = P_n + \{1, n\}$.

E3. Suppose G_1 and G_2 are graphs with exactly 1 vertex in common. That is $V(G_1) \cap V(G_2) = \{v\}$ for some v. If $T(G)$ denotes the number of spanning trees of a graph G, then show that

$$T(G_1 \cup G_2) = T(G_1)\,T(G_2)$$

Here $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Now find $T(G)$ where G is the following graph



E4. Prove the following lemma: Suppose $S \subseteq [m]$. Then,

(1) For $\sigma \in \mathrm{Perm}([m])$ such that $[m]\backslash S \subseteq [m]^\sigma$, we have that $\sigma|_S \in \mathrm{Perm}(S)$. Here $\sigma|_S : S \to [m]$ is the restriction of σ to S and $[m]^\sigma$ denotes the set of fixed points under the permutation σ.

(2) For σ as above, we get $\mathrm{sgn}(\sigma|_S) = \mathrm{sgn}(\sigma)$.

(3) The mapping $\sigma \mapsto \sigma|_S$ is a bijection between

$$\{\sigma \in \mathrm{Perm}([m]) \mid [m] \backslash S \subseteq [m]^\sigma\} \leftrightarrow \mathrm{Perm}(S).$$

E5. Recall from linear algebra the notion of the adjoint matrix. For a matrix A, we define matrix $\mathrm{adj}(A)$ as

$$\mathrm{adj}(A)_{ji} = (-1)^{i+j} \det A^{(i,j)}$$

where $A^{(i,j)}$ is the matrix obtained by A after deletion of i th row and j th column. Now let G be a graph and $L(G)$ be the Laplace matrix of G

(1) Let $v \in \mathbb{C}^n$ be the column vector with all 1 s . Verify that $L(G)v = 0$.

(2) What is $\mathrm{rank}(L(G))$? What is then the null space of $L(G)$?

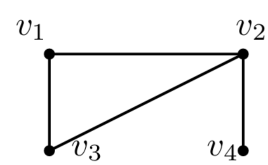
(3) What is $\det(L(G))$? What is the product $L(G) \times \mathrm{adj}(L(G))$?

(4) Conclude that $\left|\det L(G)^{(i,j)}\right|$ is the indepenent of i, j. Use this to further conclude that the proof of Kirchhoff's theorem given in the class did not depend on which row was removed from the incidence matrix.

E6. Let $T(K_n)$ be the number of spanning trees of the complete graph K_n, as defined above. Show that

$$(n-1)T(K_n) = \sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} T(K_k)\, T(K_{n-k}).$$

E7. (**Exercise to submit**)
Consider the following graph $G = (V, E)$:



(a) Determine all spanning trees of G.

(b) Compute the Laplacian matrix of G.

(c) Use Krichhoff's theorem to determine the number of spanning trees of G.